8.6.1 Fresnel Diffraction



Fig. 15 Propagation of a spherical wave

When the dimension of the diffraction object is comparable with the distance, ρ , between the light source and a small area element ds on the wave front around the diffraction object and the distance, r, between ds and the observation point, shown in figure 15, it's Fresnel diffraction. The electrical field of the wave at point P is the sum of the electrical field of all wavelets emitted from the wave front determined by the diffraction object.

$$E_{P} = -\frac{\varepsilon_{0}i}{\lambda} \iint_{\substack{available \\ wavefront}} K(\theta) \frac{e^{i[\omega t - k(\rho + r)]}}{\rho r} ds \qquad (8.33)$$

 λ, k are the wave length, wave number of the light.

$$K(\theta) = (1 + \cos \theta) / 2 \qquad (8.34)$$

Is called inclination(obliquity) factor, where θ is the angle between the normal of the ds and the r direction.

We divide the wave front onto number of angular regions rotating around the axis connecting the light source and point P. The radii of the boundaries are $r_0 + \lambda/2, r_0 + \lambda, r_0 + 3\lambda/2$ and so forth. These are the half-period zones. Waves from all points within each zone are coherent and are in phase at P.

The contribution by each zone can be obtained by the use of figure 16. The area of ds is

$$ds = \rho \quad d\phi 2\pi (\rho \sin \phi)$$

and $r^2 = \rho^2 + (\rho + r_0)^2 - 2\rho (\rho + r_0) \cos \phi$
Upon differentiation, this yields
 $2rdr = 2\rho (\rho + r_0) \sin \phi d\phi$
with ρ and r_0 held constant. Makeing use of the value of $d\phi$,

$$ds = 2\pi \frac{\rho}{\rho + r_0} r dr$$

The contribution from zone j enclosed by the boundaries $r_{j-1} = r_0 + (j-1)\lambda/2, r_j = r_0 + j\lambda/2$ $E_{\perp} = -\frac{\varepsilon_0 i}{\epsilon_0} \frac{e^{i(\omega t - k\rho)}}{k(\rho)} k(\rho) \int_{0}^{r_j} e^{-ikr} dr$

is

$$\begin{aligned}
E_{j} &= \lambda \left(\rho + r_{0}\right)^{\kappa(0_{j})} \int_{r_{j-1}}^{\infty} c^{-ikr} dr \\
&= \frac{\varepsilon_{0}}{e^{i(\omega t - k\rho)}} \frac{e^{i(\omega t - k\rho)}}{(\rho + r_{0})} k(\theta_{j}) \left[e^{-ikr_{0}} - e^{-ikr_{j-1}}\right] \\
&= \frac{\varepsilon_{0}}{e^{i(\omega t - k\rho)}} \frac{e^{i(\omega t - k\rho)}}{(\rho + r_{0})} k(\theta_{j}) e^{-ikr_{0}} \left[e^{-i\frac{2\pi}{\lambda}j\frac{\lambda}{2}} - e^{-i\frac{2\pi}{\lambda}(j-1)\frac{\lambda}{2}}\right] \\
&= (-1)^{j} \frac{2\varepsilon_{0}}{(\rho + r_{0})} \frac{e^{i(\omega t - k(\rho + r_{0}))}}{(\rho + r_{0})} k(\theta_{j}) \quad (8.35)
\end{aligned}$$



Fig. 16 Propagation of a spherical wavefront

So the contribution from adjacent zones are out of phase and tend to cancel. However, the ⁸⁻²¹ inclination factor K makes a crucial difference. As j increases, θ increases and K decreases. So that successive contributions do not completely cancel each other. The sum of the electrical fields from all m zones at P when m is add is

$$E = E_1 + E_2 + E_3 + \dots + E_m$$

= $|E_1| - |E_2| + |E_3| - \dots + |E_m|$
= $\left|\frac{E_1}{2}\right| + \left(\frac{E_1}{2}\right| - |E_2| + \left|\frac{E_3}{2}\right|\right) + \left(\frac{E_3}{2}\right| - |E_4| + \left|\frac{E_5}{2}\right|\right) + \dots + \left(\frac{E_{m-2}}{2}\right| - |E_{m-1}| + \left|\frac{E_m}{2}\right|\right) + \left|\frac{E_m}{2}\right|$

Since the quantity inside each bracket is zero, so when m is odd

$$E \approx \left| \frac{E_1}{2} \right| + \left| \frac{E_m}{2} \right| \tag{8.36}$$

When m is even

$$\begin{split} E &= E_1 + E_2 + E_3 + \dots + E_m \\ &= |E_1| - |E_2| + |E_3| - \dots - |E_m| \\ &= \left|\frac{E_1}{2}\right| + \left(\frac{E_1}{2}\right| - \left|\frac{E_2}{2}\right|\right) + \left(-\frac{|E_2|}{2}\right| + |E_3| - \left|\frac{E_4}{2}\right|\right) + \left(-\frac{|E_4|}{2}\right| + |E_5| - \left|\frac{E_6}{2}\right|\right) + \dots \left(-\frac{|E_{m-1}|}{2}\right| + |E_{m-1}| - \left|\frac{E_m}{2}\right|\right) - \left|\frac{E_m}{2}\right| \\ &\text{So when m is even} \\ E \approx \left|\frac{E_1}{2}\right| - \left|\frac{E_m}{2}\right| \end{split}$$

$$(8.37)$$

When m is very large, E_m becomes very small due to very small value of K. E becomes $E=|E_1/2|$. So the electrical field generated by the entire unobstructed wavefront is approximately equal to one half the contribution from the first zone.

For a flat circular aperture or an obstacle, the radius of the mth zone on the wavefront , R_m , can be obtained from figure 17.

$$(\rho_m + r_m) - (\rho_0 + r_0) = m\lambda/2 \qquad (8.38)$$

$$\rho_m = (\rho_0^2 + R_m^2)^{1/2} \approx \rho_0 [1 + (R_m / \rho_0)^2) = \rho_0 + \frac{R_m^2}{2\rho_0}$$

$$r_m = (r_0^2 + R_m^2)^{1/2} \approx r_0 [1 + (R_m / r_0)^2) = r_0 + \frac{R_m^2}{2r_0}$$

Put them into (8.38),

$$(\frac{1}{\rho_0} + \frac{1}{r_0}) = \frac{m\lambda}{R_m^2}$$
(8.39)



Fig 17 Relation between R_m and ther parameters

8.6.2 Circular Aperture

Envision a monochromatic spherical wave impinging on a screen containing a small hole with radius R as shown in figure 18. At point P, the electrical field, according to (8.36) and (8.37) is

$$E = \left| \frac{E_1}{2} \right| \pm \left| \frac{E_m}{2} \right|,$$

For a small hole, m is small, $|E_1| \approx |E_m|$. Then when m is even, E=0. When m is odd, E=|E₁|, which is twice the amplitude of the unobstructed wave. The irradiance is four time as large.









Fig. 19 (a) Zones in a circular aperture



When the radius of the aperture increases from very small to large, the total number of zones, m, increases, which undergoes odd, even alternatively. As a result, the irradiance at a point P on the axis will appear bright, dark alternatively.

To map the rest of the pattern, let's consider observation points along a line perpendicular to the symmetric axis, as shown in figure 19(a). Assume that at point P there are two complete zones filling the aperture and so E=0. At P₁, the second zone has been partially obscured and the third begins to show; E is no longer zero. At P2, a good fraction of the second zone is hidden,

whereas the third is even more evident. Since the contributions from the 1st and the 3rd are in ⁸⁻²⁴ phase, P₂ should be bright. Further outwards, portion of successive zones are uncovered, the irradiance undergoes a series of relative maxima and minima. Slightly beyond the geometric shadow at P₃, the first zone is only partially blocked, so the irradiance is still none zero. Further into the geometric shadow, the entire first zone is blocked and E_m is negligible since m is large, the irradiance is indeed zero.

Figure 19(b) shows the diffraction patterns for a number of holes ranging in diameter from 1mm to 4 mm as they appear on a screen 1m away. From the top left and moving right, the first four holes are so small only a fraction of the first zone is uncovered. So they are bright at the center. The sixth hole uncovers the first and second zones and is therefore black at the center. The ninth hole uncovers the first three zones and is once again bright at the center. The visible size on the screen is proportional to the size of the hole.

Let's now move the point P along the symmetry axis. When it's very close to the aperture, m, the total number of zones becomes very large according to formula (8.39), resulting small E_m and so bright at P. When P is moving away, m is getting small and undergoes alternative odd and even so the irradiance changes alternatively. When P is very far from the aperture, only small portion of first zone left. So it's always bright at P. This is a Franhoffer diffraction.

8.6.3 Circular obstacle

When a circular obstacle is inserted between a light source and a screen P, suppose the first *i* zones are blocked off. The electrical field at P is

 $E = |E_{l+1}| - |E_{l+2}| + \dots \pm |E_m| \approx \left|\frac{E_{l+1}}{2}\right| \pm \left|\frac{E_m}{2}\right|$



Fig 20 Circular Obstacle

Now $m \to \infty$, so $E \approx \left| \frac{E_{l+1}}{2} \right|$. There is a bright spot every where along the central axis except immediately behind the circular obstacle where $l \to \infty$. Figure 21 shows a diffraction pattern of a 1/8 inch bearing under the illumination of a He-Ne laser.

8.6.4 Fresnel zone plate

When either all even or all odd zones are removed, the irradiance will increase tremendously at point P. A screen that blocks light from every other half zone is called a zone plate as shown in figure 22. Suppose a zone plate passes only the first 20 odd zones and obstructs the even zones.

 $E = E_1 + E_3 + E_5 + \ldots + E_{39} \approx 20E_1$

The irradiance is almost $[20E_1/(1/2E_1)]^2=1600$ times as bright as the unobstructed wave. The radii of each zone on a zone plate can be calculated by formula (8.39).



Fig 21 Diffraction pattern of a 1/8 inch bearing



Fig 22 Zone plates, center is blocked on the left one.